KUWAIT UNIVERSITY

Department of Mathematics & Computer Science

Math 102 July 1, 2000

First Mid-term Exam Duration: 75 minutes

Calculators and mobile phones are not allowed

1. Show that $f(x) = 1 - \ln(3x)$ has an inverse. Find $f^{-1}(x)$ and state its domain and range. [4 pts]

2. Given that $f(x) = 2x^3 + 5x + 3$, find the slope of the tangent line to the graph of $f^{-1}(x)$ when $x = 10$. [3 pts]

3. Use logarithmic differentiation to find $f'(x)$, where [3 pts]

$$f(x) = \frac{\sqrt[3]{x+1}\,\sec x}{\sqrt{x}\,\sin x}$$

4. Show that: [3 pts]

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x, \qquad \text{for all } x > 0$$

5. Prove the identity: [3 pts]

$$\frac{\operatorname{sech} x}{1 - \tanh x} = \cosh x + \sinh x$$

6. Evaluate the limit: [3 pts]

$$\lim_{x \to \infty} (2^x + 3^x)^{\frac{1}{x}}$$

7. Evaluate the following integrals: [3 + 3 pts]

(a) $\displaystyle\int \frac{e^{3-x}}{1 + e^{-2x}}\, dx$

(b) $\displaystyle\int \frac{1}{x\sqrt{x^3 - 4}}\, dx$

(1) $f(x) = 1 - \ln(3x) \underset{x>0}{\Rightarrow} f'(x) = -\frac{3}{3x} = \frac{-1}{x} = \ominus \quad \forall x > 0.$

$D_f = (0, \infty)$

$R_f = (-\infty, \infty)$

f is $\searrow \Rightarrow f$ is 1-1 $\Rightarrow f^{-1}(x)$ exist.

let $y = f(x) \Rightarrow y = 1 - \ln(3x) \Rightarrow \ln(3x) = 1 - y$

$3x = e^{1-y} \Rightarrow x = \frac{1}{3} e^{1-y}$

$f^{-1}(x) = \frac{1}{3} e^{1-x}, \quad D_{f^{-1}} = (-\infty, \infty), \; R_{f^{-1}} = (0, \infty).$

(2) $f(x) = 2x^3 + 5x + 3 \Rightarrow f'(x) = 6x^2 + 5$

$x = 10 \in$ Graph of $f^{-1} \Rightarrow y = 10 \in$ Graph of f

$\Rightarrow 10 = 2x^3 + 5x + 3$

$\Rightarrow x = 1$

$\Rightarrow x = 10 \in f^{-1} \Rightarrow y = 1 \in f^{-1}$

$p(10, 1) \in f^{-1}.$

$$\frac{d}{dx}\left(f^{-1}(x)\right)\Bigg|_{\substack{x=10\\y=1}} = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(1)} = \frac{1}{11}$$

(3) let $y = \dfrac{(x+1)^{1/3} \cdot (\sec x)}{x^{1/2} \cdot \sin x} \Rightarrow \ln y = \frac{1}{3}\ln(x+1) + \ln(\sec x) - \frac{1}{2}\ln x - \ln \sin x$

$$D_x \quad \frac{y'}{y} = \frac{1}{3(x+1)} + \frac{\sec x \tan x}{\sec x} - \frac{1}{2x} - \frac{\cos x}{\sin x}$$

$$y' = \left(\frac{1}{3(x+1)} + \tan x - \frac{1}{2x} - \cot x\right) y$$

where $y = \dfrac{\sqrt[3]{x+1}\, \sec x}{\sqrt{x}\, \sin x}.$

(4) let $y = \tan^{-1}\left(\frac{1}{x}\right)$, $x > 0 \Rightarrow y \in [0, \pi/2)$

$$\Rightarrow \frac{1}{x} = \tan y \Rightarrow x = \cot y \Rightarrow \cot^{-1} x = y$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x .$$

..أو let $f(x) = \tan^{-1}\left(\frac{1}{x}\right) - \cot^{-1} x$

$$f'(x) = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2} - \frac{-1}{1+x^2} = \frac{-1}{x^2+1} + \frac{1}{1+x^2} = 0$$

$\Rightarrow f(x) = \text{Const.}$ at $x = 1$

$$\Rightarrow \tan^{-1} 1 - \cot^{-1} 1 = c. \Rightarrow c = \frac{\pi}{4} - \frac{\pi}{4} = 0$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) - \cot^{-1} x = 0 \Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x \quad \#$$

(5) $$\text{L.H.S} = \frac{\operatorname{sech} x}{1 - \tanh x} = \frac{\frac{1}{\cosh x}}{1 - \frac{\sinh x}{\cosh x}} = \frac{\cancel{\cosh x} \cdot \frac{1}{\cancel{\cosh x}}}{\cosh x - \sinh x}$$

$$= \frac{1}{\cosh x - \sinh x} = \frac{1}{\cosh x - \sinh x} \cdot \frac{\cosh x + \sinh x}{\cosh x + \sinh x} = \frac{\cosh x + \sinh x}{\cosh^2 x - \sinh^2 x}$$

$$= \frac{\cosh x + \sinh x}{1} = \cosh x + \sinh x = \text{R.H.S.}$$

(6) $$L = \lim_{x \to \infty} \left(2^x + 3^x\right)^{\frac{1}{x}} \longrightarrow \infty^0 \text{ - form}$$

$$y = \left(2^x + 3^x\right)^{\frac{1}{x}} \Rightarrow \ln y = \frac{\ln(2^x + 3^x)}{x} \longrightarrow \frac{\infty}{\infty} \text{-form}$$

Using L'Hopital Rule, we have

$$(\ln y) \longrightarrow \frac{2^x \ln 2 + 3^x \ln 3}{2^x + 3^x} = \frac{\cancel{3^x}\left(\left(\frac{2}{3}\right)^x \ln 2 + \ln 3\right)}{\cancel{3^x}\left(\left(\frac{2}{3}\right)^x + 1\right)}$$

$$\xrightarrow[x \to \infty]{} \frac{0 + \ln 3}{0 + 1} = \ln 3$$

$$\lim_{x \to \infty} \left(2^x + 3^x\right)^{\frac{1}{x}} = e^{\ln 3} = 3 \quad \#$$

⑦ (a) $\int \frac{e^{3-x}}{1+e^{-2x}}\,dx \quad \left(e^{-x} = u \Longrightarrow -e^{-x}dx = du \right)$

$$I = \int \frac{e^{3}\cdot e^{-x}\,dx}{1+(e^{-x})^2} \sim e^{3}\int \frac{-du}{1+u^2}$$

$$= -e^{3}\left(\tan^{-1}u\right) + c = -e^{3}\tan^{-1}(e^{-x}) + c.$$

(b) $I = \int \frac{1}{x\sqrt{x^3-4}}\,dx$

$$u^2 = x^3 \Longrightarrow x = u^{3/2} \Longrightarrow dx = \frac{3}{2}u^{\frac{1}{2}}du$$

$$I = \int \frac{1}{u^{3/2}\sqrt{u^2-4}}\cdot\frac{3}{2}u^{1/2}du$$

$$= \frac{3}{2}\int \frac{1}{u\sqrt{u^2-4}}\,du$$

$$= \frac{3}{2}\left\{\frac{1}{2}\sec^{-1}\frac{u}{2}\right\} + c.$$

$$= \frac{3}{4}\sec^{-1}\left(\frac{x^{3/2}}{2}\right) + c \;\#$$

Another solution, let $u^2 = x^3 - 4 \Longrightarrow 2u\,du = 3x^2dx \Longrightarrow x^2dx = \frac{2}{3}u\,du$

$$I = \int \frac{x^2}{x^3\sqrt{x^3-4}}\,dx \rightarrow \int \frac{1}{(u^2+4)\cdot\cancel{u}}\cdot\frac{2\cancel{u}}{3}du = \frac{2}{3}\int \frac{1}{u^2+4}\,du$$

$$= \frac{2}{3}\left\{\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right\} + c \Longrightarrow I = \frac{1}{3}\left\{\tan^{-1}\frac{\sqrt{x^3-4}}{2}\right\} + c \;\#$$